η = dimensionles radial distance = r/a

θ = angle measured from forward stagnation point, rad.

μ = viscosity of fluid, lb./(ft.)(hr.)

π = 3.14159 . . . . . , dimensionless

ρ = density of fluid, lb./cu.ft.

ψ = dimensionless temperature or mass fraction = $T - T_o/T_\infty - T_o$ or $W - W_o/W_\infty - W_o$

Subscripts

o = at surface of sphere ($r = a$)

∞ = far from sphere ($r \to \infty$)

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Inconsistency in the Transfer Analogies

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The Colburn analogy (*3*) for heat transfer and the Chilton-Colburn analogy (*2*) for mass transfer have provided a useful framework for the correlation of transfer coefficients. However, the success of these analogies with fluid friction has not been unqualified, since different investigators working with the same system have defined different lines on a plot of j factor and friction factor vs. Reynolds number. The purpose of this discussion is to point out a fundamental inconsistency in the analogies which may explain their partial failure.

The friction factor plot upon which the analogy is graphically based represents the frictional loss in terms of number of velocity head per unit dimensionless length (L/D). It applies only to the pressure drop in tubes at a considerable distance from the inlet of the tube. Prandtl and Tietjens (*12*) have shown that the friction factor plot can be constructed from Hagen's (*6*) pressure drop data for the entire pipe (from supply tank to outlet), provided the abnormally high entrance pressure drop is subtracted. The subtraction amounts to 2.7 velocity heads in laminar flow, and 1.4 velocity heads in turbulent flow.

In each case, one velocity head represents the gain in kinetic energy achieved by the acceleration of the fluid from zero velocity in the supply tank to the mean velocity in the tube. The remainder represents the extra pressure loss associated with the establishment of the final form of the velocity profile. After calculating pressure drop in a tube with the aid of the friction factor plot, one adds the appropriate "entrance loss."

The accepted concept of no-slip at the tube wall requires a shear stress at the tube entrance approaching infinity, since the fluid near the wall must decelerate from a finite velocity to a velocity approaching zero. The work of Deissler (*4*) has shown that the friction factor does indeed approach infinity at the tube entrance and that it decreases toward a constant terminal value as the distance from the inlet increases. If the tube is sufficiently long, the integral value of the "extra" pressure drop should be equivalent to the entrance loss mentioned above.

If heat or matter is being transferred at the tube wall, an analogous situation exists. At the tube entrance, heat or matter is transferred to fluid elements closest to the tube wall through a distance approaching zero, so the transfer coefficient approaches infinity. The coefficient declines toward a constant terminal value as the tube length increases and the mean distance for transfer increases toward an asymptotic value. This aspect is discussed in considerable detail elsewhere (*8*).

LAMINAR FLOW

For the case of heat transfer with laminar flow in a short cylindrical conduit with constant wall temperature, parabolic velocity distribution established, and invariant properties, Colburn (*3*) converted Lévêque's (*9*) equation

$$\frac{h_a}{\rho c_p} = 1.615 \left(\frac{\alpha^2 \bar{u}}{DL} \right)^{1/3}$$

$$= 1.615 \left(\frac{\alpha^2}{D\theta} \right)^{1/3} \quad (1)$$

to the form

$$j = N_{St} N_{Pr}^{2/3} = 1.615\, N_{Re}^{-2/3}\, (L/D)^{-1/3} \quad (2)^*$$

Lévêque's equation predicts an infinite value for the coefficient at the start of the heated length ($L = 0$) and a continuous decrease in coefficient with increase in heated length. If heat transfer were started at the tube inlet, where the velocity distribution approaches uniformity, the heat transfer analog of Higbie's equation would apply for a very short distance, again predicting an infinite coefficient.

As discussed by Norris and Streid (*10*) it is regrettable that h_a, the arithmetic mean coefficient, was chosen for the analogy. Lévêque also showed that h_M, the coefficient based on log mean temperature difference, is defined for short tubes by the same equation as above for the coefficient based on arithmetic mean Δt.

Lévêque further showed that, unlike h_a, the log mean coefficient approaches a constant terminal value, which can be expressed in the form

$$\text{Minimum Nusselt number} = 3.66 \quad (3)$$

This terminal value is in complete accord with Graetz's (*5*) precise infinite series solution to the problem, which does not depend on the assumption of a short heated length.

Since the friction factor plot excludes the inlet behavior and gives only the terminal "coefficient" for momentum

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* The numerical constant in Colburn's paper does not agree with that given by Lévêque.

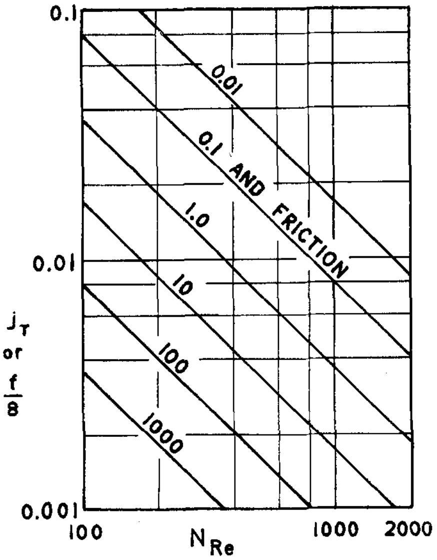

Fig. 1. Terminal j factor in the laminar range. Parameter: Prandtl or Schmidt number.

transfer well down the pipe, proper application of the analogy requires that only terminal heat or mass transfer coefficients be plotted.

The terminal j factor derived from the minimum Nusselt number is

$$j_T = 3.66\, N_{Re}^{-1} N_{Pr}^{-1/3} \qquad (4)$$

In Figure 1, this is plotted for various values of the Prandtl number. All the lines are parallel to each other and to the friction factor, with the j factor line for $N_{Pr} = 0.1$ coinciding with the friction factor line.

The fact that all the lines are parallel demonstrates the analogy in a more convincing way than the standard plot on which none of the lines is parallel to the friction factor line. If, instead of the j factor, the Stanton number is plotted as ordinate, just two lines are obtained, the friction factor line and the parallel line shown in Figure 1 for a Prandtl number of 1.0. There is no fundamental requirement that the two coincide at a Prandtl number of 1.0, since the final form of the velocity gradient is not the same as the final form of the temperature or concentration gradient. (For simplicity, the discussion has been confined to heat transfer terminology; for mass transfer, $k'/\bar{u}$ replaces N_{St}, N_{Sc} replaces N_{Pr}, and N_{Sh} replaces N_{Nu}.)

TURBULENT FLOW

Heat and mass transfer coefficients and friction factors behave qualitatively the same in turbulent flow as in laminar flow; that is, they approach infinity at the start of the transfer process and decrease toward constant terminal values with increase in fluid residence time (or increase in pipe length). This is illustrated by Deissler's (4) graphs of local Nusselt number and local friction factor vs. L/D.

With fluids of low Prandtl or Schmidt number the constant terminal coefficient is approached quite rapidly. The achievement of a constant coefficient is slower when the Prandtl number is high, as for viscous oils, or when the Schmidt number is high, as in mass transfer operations involving liquids.

Because of the relative lack of precision of analytical expressions for turbulent heat and mass transfer, a different procedure will be used to illustrate the inconsistency in the analogy.

In Figure 2, total friction factor plots are presented for the turbulent regime. These are total in the sense that 0.4 velocity heads have been added to account for the abnormally high stresses near the tube inlet which are associated with the development of the velocity profile. The curves have been terminated at the left-hand extremities in accord with the fact that a certain minimum length is required to set up the velocity profile (and to consume 0.4 velocity heads of "extra" pressure drop). The relation used to develop the cutoff point is that given by Prandtl (*11*):

$$L/D = 500{,}000\, N_{Re}^{-1} \qquad (5)$$

Just as the total friction factor increases with decreasing L/D, so do heat and mass transfer coefficients, which are total also in the sense that the abnormally high coefficient near the entry is included in the overall mean coefficient.

One may expect, then, that families of curves like those in Figure 2 will be obtained when j factors are plotted for various L/D ratios. It will be noted in Figure 2 that each line has a different slope; that is, each analytical expression for the j factor will show proportionality to a different power of the Reynolds number.

Alternatively, two experimenters working with same system but with apparatus of different length can elucidate two different values of j at the same Reynolds number. If these results are forced to conform to the friction factor curve for L/D = ∞, the difference can only be reconciled by assigning two different powers to the Prandtl or Schmidt number.

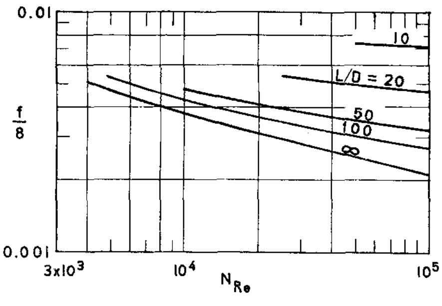

Fig. 2. Total friction factor, including entry loss, in the turbulent range.

That the magnitude of mean coefficients is dependent on L/D in turbulent flow as well as in laminar flow has been demonstrated experimentally by Kays and London (*7*). In their Figure 29, the j factor plots are similar to Figure 2 presented here.

IMPLICATIONS

Although the discussion thus far has been specific to transfer in cylindrical conduits, the implications are by no means so limited. For example, the fluid passageways in a packed bed may be considered as conduits of varying cross section. Barker (*1*) presented j factor plots for heat transfer in packed beds with data from a number of sources. The agreement among the results of different investigators is poor. A check of the source data reveals a trend in which the j factor at any given Reynolds number decreases with increase in bed depth.

These data are for heat transfer from gases to solids. The same sort of dependence on depth can be expected for mass transfer involving gases. Even more dramatic variation can be expected for the processes of heat and mass transfer with liquids. In view of the present strong trend toward extreme miniaturization of experimental equipment, it is essential that L/D or fluid residence time be taken into account when attempting to apply the analogies, whether the fluid flow is laminar or turbulent.

NOTATION

c_p = specific heat, B.t.u./lb.
D = diameter (of conduit or particle), ft.
$\mathfrak{D}$ = diffusivity, sq. ft./hr.
f = friction factor, $(\Delta P/\rho)/(L/D)(\bar{u}^2/2g_c)$
g_c = Newton's law factor, 4.17 (10^8) (lb.$_m$) (ft.)/(lb.$_f$) (hr.^{2})
h = heat transfer coefficient, B.t.u./(hr.) (sq. ft.) (°F.)
j = j factor, $N_{St}\, N_{Pr}^{2/3}$ for heat transfer, $(k'/\bar{u})\, N_{Sc}^{2/3}$ for mass transfer
k = thermal conductivity, (B.t.u.)(ft.)/(hr.) (sq.ft.) (°F.)
k' = mass transfer coefficient, ft./hr.
L = length, ft.
N_{Nu} = Nusselt number, hD/k
N_{Pr} = Prandtl number, $c_p\mu/k$
N_{Re} = Reynolds number, $D\bar{u}\rho/\mu$

N_{Sc} = Schmidt number, $\mu/\rho\mathfrak{D}$
N_{Sh} = Sherwood number, $k'D/\mathfrak{D}$
N_{St} = Stanton number, $h/c_p u \rho$
P = pressure, lb./sq.ft.
$\bar{u}$ = mean velocity, ft./hr.

Greek Letters

α = thermal diffusivity, $k/c_p\rho$, sq. ft./hr.
θ = mean residence time, hr.
μ = viscosity, lb./(ft.)(hr.)
ρ = density, lb./cu.ft.

Subscripts

a = arithmetic mean
M = logarithmic mean
T = terminal value

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The Composition Dependence of Effective Diffusivities for Electrolyte Penetration in Ion Exchange Resins

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During a recent analysis of a particle diffusion controlled ion exclusion process (1) in the system Dowex 50Wx8(Na^+ form)–water–sodium chloride–glycerol, literature values for the effective resin phase diffusivities of sodium chloride and glycerol (*4*) in this system were used to calculate theoretical column effluent curves. Because the resulting theoretical curves were in considerable disagreement with experimental effluent curves, a short experimental study was performed to check the reported diffusivities (in particular, sodium chloride). New sodium chloride diffusivity values, which gave excellent agreement between the theoretical and experimental sodium chloride effluent curves, were obtained. The purpose of this note is to present these new values, to explain the discrepancy between these new values and those reported earlier, and to indicate a method by which the new data may be correlated.

METHOD FOR MEASURING RESIN PHASE SODIUM CHLORIDE DIFFUSIVITIES

Effective resin phase diffusivities for sodium chloride were measured by the same technique used in the earlier study (*4*) with one important difference, which will be mentioned later. Briefly, the method involves saturating a bed of resin with a sodium chloride-glycerol–water solution of known composition, eluting the bed with a sodium chloride-free solution for a definite interval of time, extracting the sodium chloride remaining in the resin with water, and titrating the extract with standard silver nitrate solution to determine the total sodium chloride content. This procedure is repeated (using the same saturating solution) for several different elution times and a plot of the amount of sodium chloride remaining in the resin vs. elution time is prepared. By matching this plot with the following theoretical result, we obtain the sodium chloride diffusivity

$$\rho_i/\rho_i{}^{\circ} = (6/\pi^2) \sum_{m=1}^{\infty} (1/m^2) \exp\,(-m^2\pi^2 D_i\, t/R^2) \qquad (1)$$

The entire series of runs may be repeated for different saturating solution compositions to give information regarding the variation of the sodium chloride diffusivity with composition.

Inherent in the familiar theoretical Equation (1) are a number of assumptions: (a) pressure diffusion (associated with swelling and shrinking) may be neglected, (b) forced diffusion (for example due to electrical gradients) need not be considered explicitly, (c) the mass average velocity of the resin phase is zero, (d) the sodium chloride concentration over all particle surfaces is zero for $t > 0$, (e) there occurs no coupling between the sodium chloride diffusion flux and other diffusion fluxes, and (f) D_i is constant throughout the elution period. Assumption (b) is known (*2*) to be reasonably good for electrolyte diffusion in ion exchange resins where the coion concentration is considerably smaller than the counterion concentration, as in the present case. Assumptions (a), (c), (d), and (e) are probably reasonable, but have not yet been thoroughly checked. With regard to the last assumption, it is to be expected that the D_i may be strongly dependent on the composition of the internal solution of the resin phase, primarily through the dependence of the D_i on the viscosity of the internal solution, that is, through Nernst-Einstein effects (electrical effects, though present, do not lead to strong composition dependence).

PREVIOUS STUDIES

The resin phase sodium chloride and glycerol diffusivities reported in the literature (*4*) reflect the neglect of the above mentioned composition dependence. In this previous work, saturating